Europ. J. Combinatorics (2002) **23**, 631–633
doi:10.1006/eujc.2002.0567
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On a Lemma of Gromov and the Entropy of a Graph

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If G is a finite, directed, simple and irreducible graph, deletion of an edge makes the entropy decrease. We give a proof of this fact that avoids the Perron–Frobenius theorem and makes use of a technique developed by Gromov.

1. INTRODUCTION

Let G be a finite, directed, simple graph. In the terminology of [3, p. 43], G is a graph whose adjacency matrix contains only 0's and 1's. If G is irreducible, its entropy decreases if an edge is deleted. This is a standard fact in the theory of symbolic dynamical systems and is proved by means of the Perron–Frobenius theory [3, p. 123]. In [2] Gromov proved a similar entropic inequality for a more general class of symbolic systems, but his proof does not apply directly to our case; see also [1]. In this paper we give a Gromov type proof of the entropic inequality for irreducible graphs. It does not make use of the Perron–Frobenius theory. From the entropic inequality proved here, the entropic inequality for irreducible sofic dynamical systems [3, p. 124], the Hedlund–Coven–Paul Theorem [3, p. 268, 269], and the Garden of Eden Theorem for irreducible symbolic systems of finite type [1] follows easily.

Let G be a finite, simple, directed graph. A *path* in G is a sequence $a_1a_2\ldots a_m$ of vertices such that there is an edge starting in a_i and terminating in a_{i+1} for $i = 1, 2, \ldots, m-1$. Define $B_m(G)$ as the set of all paths in G of length m. The entropy $h(G)$ of G is defined as

$$h(G) \stackrel{\text{def}}{=} \lim_{m\to\infty} \frac{\log|B_m(G)|}{m}.$$

Such a limit always exists [3, p. 104]. A graph G is *irreducible* when, given any ordered pair of vertices a and b, there exists a path from a to b. An *arc* in G is a simple path, i.e., a path in which the vertices are all distinct. A *cycle* is a simple closed path, i.e., a path $a_1a_2a_3\ldots a_n$ such that $a_1 = a_n$ and $a_i \neq a_j$ if $\{i, j\} \neq \{1, n\}$. Given an arbitrary path $p = a_1a_2a_3\ldots a_n$, we can decompose it into cycles as follows. Let a_{i_1} be the first vertex such that the vertices $a_1, a_2 \ldots a_{i_1-1}$ are all distinct and $a_{i_1} = a_{j_1}$ for a $j_1 < i_1$; then $c_1 = a_{j_1}a_{j_1+1}\ldots a_{i_1}$ is the *first cycle of the path*. Next, consider the first vertex a_{i_2}, with $i_2 > i_1$, such that the vertices $a_{i_1}a_{i_1+1}\ldots a_{i_2-1}$ are all distinct and $a_{i_2} = a_{j_2}$ for a $j_2 \in \{i_1, i_1+1, \ldots, i_2-1\}$; then $c_2 = a_{j_2}a_{j_2+1}\ldots a_{i_2}$ is the *second cycle of the path*. Continuing in this way we obtain a canonical decomposition $p = r_1c_1r_2c_2\ldots r_kc_kr_{k+1}$ where $c_1, c_2, \ldots, c_k$ are cycles and $r_1, r_2, \ldots, r_{k+1}$ are (possibly empty) simple arcs. Clearly if r_j is empty then c_{j-1} and c_j have a vertex in common.

LEMMA. *Let G be an irreducible graph and ab an edge in G. Then there exists n such that if $p = a_1a_2\ldots a_n$ is a path in G of length n then there exists a path $p' = a_1a'_2\ldots a'_{n-1}a_n$ containing ab.*

PROOF. Irreducibility implies the existence of a path q starting at a_1, terminating in a_1 and containing ab. If n is large enough then in the canonical decomposition $p = r_1c_1r_2c_2\ldots r_kc_k r_{k+1}$ of p there exists a cycle c that is repeated many times. If the length of c is l, the length of

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q is m and the cycle c is repeated at least m times, then we may delete the first $m-1$ copies of c and add $l-1$ copies of q at the beginning obtaining the desired path. □

We now prove the fundamental entropic inequality using Gromov's technique.

THEOREM. *If G is an irreducible graph and H is obtained from G by deleting an edge ab then $h(H) < h(G)$.*

PROOF. Let n be the integer of the lemma. We first show that if we define

$$\alpha = \frac{1}{|B_n(G)|}$$

then we have, for $k = 1, 2, \ldots$

$$|B_{kn}(H)| \leq (1-\alpha)^k |B_{kn}(G)|. \tag{$*$}$$

Clearly, $|B_{n(k-1)}(G)| \geq \alpha \cdot |B_{kn}(G)|$. In what follows a path p of length kn will be represented as the concatenation of paths of length n, i.e., in the form $p = p_1 p_2 \ldots p_k$, where p_h is a path of length n for $h = 1, 2, \ldots, k$. Define π_h as the set of all $p \in B_{kn}(G)$ such that p_h contains the edge ab. By the lemma for any $p' \in B_{(k-1)n}(G)$ there exists a path $qp' \in B_{kn}(G)$ such that q contains ab. Then $|\pi_1| \geq |B_{n(k-1)}(G)|$. Therefore $|B_{nk}(G) \setminus \pi_1| \leq (1-\alpha)|B_{nk}(G)|$.

Then define $\bar{B}^h_{nk}(G) = B_{nk}(G) \setminus \bigcup_{l=1}^{h-1} \pi_l$, $C^h = \{p \in \bar{B}^h_{nk}(G) : p_h \text{ contains } ab\}$ and $\bar{D}^h$ as the set of all couples of paths (q_1, q_2) such that $q_1 \in B_{n(h-1)}(G)$, $q_2 \in B_{n(k-h)}(G)$ and there exists $q \in B_n(G)$ such that $q_1 q q_2 \in \bar{B}^h_{kn}(G)$. By the lemma, for any $(q_1, q_2) \in \bar{D}^h$ there exists $p = q_1 q' q_2 \in B_{kn}(G)$ such that q' contains ab; thus $p \in C^h$. Therefore we have, again, $|C^h| \geq |\bar{D}^h| \geq \alpha \cdot |\bar{B}^h_{kn}(G)|$ so that

$$\begin{aligned}\left| B_{kn}(G) \setminus \bigcup_{l=1}^{h} \pi_l \right| &= \left| \left(B_{kn}(G) \setminus \bigcup_{l=1}^{h-1} \pi_l \right) \setminus \pi_h \right| = |\bar{B}^h_{kn}(G) \setminus \pi_h| = |\bar{B}^h_{kn}(G) \setminus C^h| \\ &\leq (1-\alpha)|\bar{B}^h_{kn}(G)| \leq (1-\alpha)^h |B_{kn}(G)|.\end{aligned}$$

The last inequality follows by an obvious inductive argument on h. Then for $h = m$ we obtain ($*$). Taking logarithms,

$$\frac{\log |B_{kn}(H)|}{kn} \leq \frac{\log(1-\alpha)}{n} + \frac{\log |B_{kn}(G)|}{kn}$$

and therefore letting $k \to \infty$ we have

$$h(H) \leq \frac{\log(1-\alpha)}{n} + h(G) < h(G). \qquad \square$$

ACKNOWLEDGEMENTS

I am grateful to Tullio Ceccherini–Silberstein for discussions.

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Received 16 June 2001 and accepted 17 February 2002

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